Logic programming and the intuitionistic sequent calculus

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1 Introduction

The calculus of sequents, formulated by G. Gentzen, is a system of rules for deriving expressions, called sequents, of the form

(1) $$\Gamma \rightarrow \Delta$$

where Γ and Δ are sequences (possibly empty) of predicate logic formulas. The interpretation of (1) adopted here is the one familiar from the special case of clauses (in which all formulas in Γ and Δ are atomic): if all formulas in Γ are true, then at least one formula in Δ is true. If Δ is empty, (1) states that not all formulas in Γ are true; if Γ is empty, (1) states that at least one formula in Δ is true. If Γ and Δ are both empty, (1) is the empty sequent, interpreted as a logically false statement. Thus (1) is a special notation for a predicate logic formula which could be written using the standard connectives. On another common reading, (1) is interpreted as stating that the disjunction of the formulas in Δ is a logical consequence of the formulas in Γ.

A current project at SICS, headed by Dan Sahlin, investigates the possibility of basing a logic programming system on the rules of the so-called cut-free intuitionistic (or constructive) sequent calculus. This calculus, which is also due to Gentzen, is a system for deriving, in a constructively valid way, sequents for which Δ contains exactly one formula. In a logic programming application, Γ will contain the formulas constituting (the logical content of) a program, while Δ will consist of a query B put to the program. To construct a viable logic programming system on this basis, a class of program statements and queries must be chosen for which a reasonably efficient proof procedure can be implemented. In particular this involves finding a way of dealing with the computationally difficult equality relation.

The purpose of the present report is to present some basic logical results and observations relevant to the construction of such a system. In order to make all the considerations involved fully accessible to the non-expert, several basic concepts will be defined and discussed: Kripke models, the Heyting interpretation, invertibility of rules.

The proofs are adaptations of standard methods and results to the present context. (See in particular [4].)

2 Formulas, rules, and derivations

The language used will be that of predicate logic with equality. Two points should be noted: the use of the constant $\perp$ (interpreted as a logically false statement) as a logical primitive instead of negation, and the separation of variables into those which are only used as bound variables (called variables) and those which are only used as free variables (called parameters). The letters x,y,z will be used for variables and α,β for parameters. Thus we have *terms* which are either parameters or individual constants or composite terms $f(t_1,..,t_n)$ where $t_1,..,t_n$ are terms and f is an n-place function symbol. Individual constants will also be regarded as 0-place function symbols. The letters s,t,u,v will be used to denote terms. (Note that variables are not terms.) The *formulas* are either atomic or composite. Atomic formulas are the special atomic formula $\perp$ (falsum or the absurdity) and $p(t_1,..,t_n)$ where p is an n-place predicate symbol. We also admit 0-place predicate symbols, so that p is an atomic formula for every 0-place predicate symbol p. Composite formulas are $(A\supset B)$, $(A\vee B)$, $(A\&B)$, $\forall xA(x/\alpha)$, $\exists xA(x/\alpha)$. Here $A(x/\alpha)$ stands for the result of substituting x for every occurrence of α in A. When this notation is used it is presupposed that α does not occur in A within the scope of any quantifier $\forall x$ or $\exists x$.The expressions $A(t/\alpha)$ and $u(t/\alpha)$ are interpreted similarly. The negation $\neg A$ is defined as an abbreviation of $(A\supset \perp)$, and the equivalence $(A\equiv B)$ stands for $((A\supset B)\&(B\supset A))$. Among the predicate symbols is the identity symbol =. Equalities are written in the usual way as s=t. The letters A,B,C,D will be used to denote formulas. Parentheses will be omitted in accordance with the following conventions: 1) the outermost parentheses are omitted when formulas occur in isolation; 2) association to the right is used for $\vee$ and &; 3) $\vee$ and & bind harder than $\supset$ and $\equiv$. Thus e.g. $A\vee B\supset C\&D\&E$ stands for $((A\vee B)\supset(C\&(D\&E)))$, whereas $A\&B\vee C$ is undefined.

The parameter α is often suppressed in talking about formulas, so that one writes simply $\forall xA(x)$ or $A(s)$ instead of $\forall xA(x/\alpha)$ or $A(s/\alpha)$. Thus e.g. the identity rule (see p.4) would often be written

$$\frac{s=t, A(s), \Gamma \rightarrow B}{s=t, A(t), \Gamma \rightarrow B}$$

This convenient notation will be used in the following when no confusion is likely to result. It must be kept in mind when this notation is used that, for example, A(s) need not be the result of substituting s for t in A(t).

For purposes of comparison, the cut-free sequent calculus is presented below in both its intuitionistic and its classical form. The intuitionistic system will be referred to as GI, the classical system as GC. The system GF - taken as basic in the project to which this report pertains - which differs from GI in the identity rules is presented in §4, and a variant of the intuitionistic sequent calculus in which more than one formula is allowed in the consequent is described in §6.

Formal proofs in the sequent calculus are trees rooted in the conclusion, with axioms as tips and every other node related to its immediate predecessor(s) by one of the inference rules. The rules are all clearly logically valid in the sense that the conclusion of a rule is logically valid if the same is true of the premisses. For the moment this observation is made only with respect to classical, i.e. standard logical validity. The concept of constructive or intuitionistic logical validity will be introduced below.

There are two logical rules for each of the connectives and quantifiers: one for introducing the connective or quantifier on the left side of a sequent, i.e. in the antecedent, and one for introducing it on the right side, in the consequent. The rules are accordingly named &→, →& etc. The intuitionistic →∨-rule, it will be noted, has two forms. The formula introduced in the antecedent or consequent of the conclusion when a rule is applied is called the *principal formula* of that application.

All systematic procedures for finding a proof of a given sequent in the sequent calculus proceed by constructing a (would-be) proof tree in a bottom-up fashion.[1] It will be seen that the problems associated with such procedures in the intuitionistic and in the classical case are partly different.

To reduce verbosity, the axioms and rules are formulated below with the exhibited formulas leftmost in the antecedent. The rules are to be understood, however, as covering every permutation of the formulas in the antecedent. For example, &→ covers every step of the form

$$\frac{\Gamma \to C}{\Gamma' \to C}$$

[1] See the pioneering paper [7].

where Γ is a permutation of A,B,Δ and Γ' is a permutation of A&B,Δ. Similarly in arguments about sequents. Another way of putting this is that Γ and Δ stand for "multisets", i.e. structures of the type obtained by identifying a list with its permutations. This latter interpretation is not always appropriate, however. Consider the proof

$$\dfrac{\dfrac{p,q,r,p,q,r \rightarrow r \qquad p,q,r,p,q,r \rightarrow q}{p,q,r,p,q,r \rightarrow r\&q}}{p\&q,r,p,q,r \rightarrow r\&q}$$

If we regard the sequents as multisets, the conjunction p&q in the conclusion cannot be associated with any particular occurrences of p and q in the premiss, and similarly for r and q in the axioms. Generally speaking, we cannot follow an occurrence of a formula in the conclusion upwards through the proof. To introduce such an association of formula occurrences is to impose extra structure on the proof, structure which can contain valuable information. I will have no occasion to consider such *analyzed proofs* in this report, so sequents can be regarded as multisets in the following.

The classical and intuitionistic systems formulated below differ in that the sequents of the classical system may contain several formulas in the consequent, whereas the intuitionistic sequents have at most one. The differences in the formulations of $\rightarrow\exists$ and $\rightarrow\vee$ follow from this, and there is also a difference in the formulation of $\supset\rightarrow$. The reasons for these differences between the two systems will be explained in §§ 3,5,6.

The intuitionistic system GI

Logical axioms:

$\bot,\Gamma\to C$

$B,\Gamma\to B$

Logical rules:

$$\frac{A,B,\Gamma\to C}{A\&B,\Gamma\to C} \qquad \frac{\Gamma\to A \quad \Gamma\to B}{\Gamma\to A\&B}$$

$$\frac{A,\Gamma\to C \quad B,\Gamma\to C}{A\vee B,\Gamma\to C} \qquad \frac{\Gamma\to A}{\Gamma\to A\vee B} \quad \frac{\Gamma\to B}{\Gamma\to A\vee B}$$

$$\frac{A\supset B,\Gamma\to A \quad B,\Gamma\to C}{A\supset B,\Gamma\to C} \qquad \frac{A,\Gamma\to B}{\Gamma\to A\supset B}$$

$$\frac{A,\Gamma\to B}{\exists xA(x/\alpha),\Gamma\to B}\ (*) \qquad \frac{\Gamma\to A(t/\alpha)}{\Gamma\to\exists xA(x/\alpha)}$$

$$\frac{\forall xA(x/\alpha),A(t/\alpha),\Gamma\to B}{\forall xA(x/\alpha),\Gamma\to B} \qquad \frac{\Gamma\to A}{\Gamma\to\forall xA(x/\alpha)}\ (*)$$

(*) Restriction on the rules $\exists\rightarrow$ and $\rightarrow\forall$: the parameter α must not occur in B or in any formula in Γ.

Identity axioms:

$$\Gamma\rightarrow t=t$$

Identity rules:

$$\frac{s=t, A(s/\alpha), \Gamma\rightarrow B}{s=t, A(t/\alpha), \Gamma\rightarrow B}$$

$$\frac{s=t, A(t/\alpha), \Gamma\rightarrow B}{s=t, A(s/\alpha), \Gamma\rightarrow B}$$

$$\frac{s=t, u(s/\alpha)=u(t/\alpha), \Gamma\rightarrow B}{s=t, \Gamma\rightarrow B}$$

$$\frac{s=t, u(t/\alpha)=u(s/\alpha), \Gamma\rightarrow B}{s=t, \Gamma\rightarrow B}$$

The classical system GC

Logical axioms:

$$\bot,\Gamma\rightarrow\Delta$$

$$B,\Gamma\rightarrow\Delta,B$$

Logical rules:

$$\frac{A,B,\Gamma\rightarrow\Delta}{A\&B,\Gamma\rightarrow\Delta} \qquad \frac{\Gamma\rightarrow\Delta,A \qquad \Gamma\rightarrow\Delta,B}{\Gamma\rightarrow\Delta,A\&B}$$

$$\frac{A,\Gamma\rightarrow\Delta \qquad B,\Gamma\rightarrow\Delta}{A\vee B,\Gamma\rightarrow\Delta} \qquad \frac{\Gamma\rightarrow\Delta,A,B}{\Gamma\rightarrow\Delta,A\vee B}$$

$$\frac{\Gamma\rightarrow\Delta,A \qquad B,\Gamma\rightarrow\Delta}{A\supset B,\Gamma\rightarrow\Delta} \qquad \frac{A,\Gamma\rightarrow\Delta,B}{\Gamma\rightarrow\Delta,A\supset B}$$

$$\frac{A,\Gamma\rightarrow\Delta}{\exists xA(x/\alpha),\Gamma\rightarrow\Delta}\ (*) \qquad \frac{\Gamma\rightarrow\Delta,A(t/\alpha),\exists xA(x/\alpha)}{\Gamma\rightarrow\Delta,\exists xA(x/\alpha)}$$

$$\frac{\forall x A(x/\alpha), A(t/\alpha), \Gamma \to \Delta}{\forall x A(x/\alpha), \Gamma \to \Delta}$$

$$\frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, \forall x A(x/\alpha)} \ (*)$$

(*) Restriction on the rules $\exists\to$ and $\to\forall$: the parameter α must not occur in any formula in Γ or Δ.

Identity axioms:

$$\Gamma \to \Delta, t=t$$

Identity rules:

$$\frac{s=t, A(s/\alpha), \Gamma \to \Delta}{s=t, A(t/\alpha), \Gamma \to \Delta} \qquad \text{A atomic}$$

$$\frac{s=t, A(t/\alpha), \Gamma \to \Delta}{s=t, A(s/\alpha), \Gamma \to \Delta} \qquad \text{A atomic}$$

$$\frac{s=t, u(s/\alpha)=u(t/\alpha), \Gamma \to \Delta}{s=t, \Gamma \to \Delta}$$

$$\frac{s=t,u(t/\alpha)=u(s/\alpha),\Gamma\rightarrow\Delta}{s=t,\Gamma\rightarrow\Delta}$$

3 Contraction

The sequent calculus was originally formulated with a contraction rule of the form

$$\frac{A,A,\Gamma\rightarrow B}{A,\Gamma\rightarrow B}$$

(with a similar rule for the consequent in the classical system). When proofs are constructed in bottom-up fashion, use of the contraction rule means that a copy is made of the principal formula before it is "dissolved" (as it appears when the rules are applied backwards) so that essential information will not be lost.

The rules given above do not include any contraction rule. Instead the unavoidable uses of contraction have been incorporated into the formulation of the rules $\supset\rightarrow$ (in the intuitionistic case only), $\rightarrow\exists$ (in the classical case only), $\forall\rightarrow$ and the identity rules. Why these particular rules need contraction, and the reason for the difference between the classical and the intuitionistic case, will be commented on below in §5.

Contraction is responsible for the lack of any computable bound on the size of a proof of a given sequent. If no use is made of contraction - i.e. if no contraction rule is used, and the rules $\supset\rightarrow$, $\rightarrow\exists$, $\forall\rightarrow$ are formulated without any repetition of the principal formula - a logical system results which is a decidable fragment of standard (classical or intuitionistic) logic. For semantic and proof-theoretic studies of this fragment, see [2], [8], [10], [12].

Another decidable fragment of the calculus is the propositional system obtained by leaving out the rules for $\forall$ and $\exists$. A decision procedure for the propositional intuitionistic sequent calculus based on an analysis of the use of contraction is given in [5].

The following examples show the necessity of using contraction in the indicated rules. Let $f^{n+1}(t)$ stand for the term f(f(...f(t))) with n+1 occurrences of f. The use of contraction in the identity rules is necessary since, as is easily seen, n applications of contraction must be made (i.e. the formula $\alpha=f(\alpha)$ must be used n+1 times) in proving either of the sequents

$$\alpha=f(\alpha),p(\alpha) \rightarrow p(f^{n+1}(\alpha))$$

$$\alpha=f(\alpha) \rightarrow \alpha=f^{n+1}(\alpha)$$

The need for contraction in applications of the $\forall\!\rightarrow$ rule is shown by the sequent

$$\forall x(p(x)\supset p(f(x))),p(\alpha) \rightarrow p(f^{n+1}(\alpha))$$

which also requires n contractions. In the classical system, n contractions in applying $\rightarrow\!\exists$ are needed to prove

$$q(\alpha),\neg q(f^{n+1}(\alpha)) \rightarrow \exists x(q(x)\&\neg q(f(x))$$

The need for $\supset\!\rightarrow$ contraction in the intuitionistic system, finally, emerges when a proof is sought of

$$\neg\neg(\neg p_1 \vee\ldots\vee \neg p_n \ \vee \ (p_1\&\ldots\&p_n))$$

This formula is valid in minimal logic, i.e. the absurdity axiom is not needed for its proof. In proving it, the implication obtained by deleting the first negation sign must be used n+1 times.

4 The system GF

In order to improve the prospects for a computationally feasible identity theory, [6] introduces the free interpretation of terms. That is, the universe is assumed to be freely generated from some set by the operations for which function symbols occur in the language. In the resulting system, which will be called GF, the identity rules of GI and GC are replaced by the following (stronger) rules and axioms.

Axioms:

$\Gamma\rightarrow t=t$

$f(s_1,..s_n)=g(t_1,..t_m),\Gamma\rightarrow A$ $\quad$ $m,n\geq 0$ $\quad$ for different function symbols f,g

α=s[α],$\Gamma \rightarrow$A

s[α]=α,$\Gamma \rightarrow$A

In the last two axioms, s[α] is a term which properly contains the parameter α.

Rules:

$$\frac{s_1=t_1,..s_n=t_n,\Gamma \rightarrow A}{f(s_1,..s_n)=f(t_1,..t_n),\Gamma \rightarrow A}$$

$$\frac{\Gamma(s/\alpha) \rightarrow A(s/\alpha)}{\alpha=s,\Gamma \rightarrow A}$$

$$\frac{\Gamma(s/\alpha) \rightarrow A(s/\alpha)}{s=\alpha,\Gamma \rightarrow A}$$

In the last two rules, which will be called replacement rules, s is a term which does not contain the parameter α. The first rule has a special case n=0 in which an equation e=e, where e is an individual constant, is introduced in the antecedent. (Corresponding to the special case of the replacement rules in which s is α.)

The axioms except $\Gamma \rightarrow$t=t, and the first of the new rules (expressing that all operations are injective) do not hold in GI, unlike the replacement rules. The computationally pleasant aspect of these rules is that none of them incorporates contraction: thus formulas s=t are eliminated once and for all when a proof is sought. That no contraction is needed is clear since these three rules are all *invertible* (in the case of the last two rules because of the restriction that α does not occur in s). Invertibility is defined and discussed in §5.

Since being freely generated (in the standard algebraic sense) from some set is not in fact a first-order property of structures, the above axioms and rules necessarily hold for a wider class of structures, which I will call *quasi-free*. Thus a structure is quasi-free if the functions are injective and have pairwise disjoint ranges, and no sequence of applications of functions can lead from an individual a to a.[2]

In order to establish that the system GF is an extension of intuitionistic logic as ordinarily formulated, we need to show that contraction holds as a derived rule - i.e. if $A,A,\Gamma\rightarrow B$ is provable then so is $A,\Gamma\rightarrow B$ - and that the ordinary identity rules (those of GI and GC) hold as derived rules. To justify the proof procedure in [6] we must also show that all rules except $\supset\rightarrow$ and $\rightarrow\exists$ are invertible (see §5). These properties follow easily from the completeness theorem for GF with respect to Kripke semantics to be given below.

5 Invertibility

That the rules of the sequent calculus are logically valid means that the conclusion in an application of a rule is logically valid (classically or intuitionistically) if the same is true of the premiss or premisses. A rule is called *invertible* if the converse holds, i.e. if the logical validity of the conclusion implies the logical validity of the premisses. Invertibility as here defined is thus a semantic property of rules, i.e. it is defined solely in terms of logical validity. The discussion that follows could equally be couched in terms of a syntactic concept of invertibility relative to a system of rules, a rule then being called invertible if the premisses are provable whenever the conclusion is.

Invertibility has large consequences for the bottom-up construction of derivations. Suppose we are seeking to formulate a complete system of rules for the sequent calculus, i.e. a system such that every logically valid sequent is provable. If the system is complete it will always be possible to construct a proof of a valid sequent by means of some kind of bottom-up procedure. If a rule is invertible we know that the task of proving the conclusion can always be reduced to that of proving the premisses. If, on the other hand, a rule is not invertible, the corresponding reduction may lead to an impossible task.

As a consequence, an invertible rule is always easier to deal with than a non-invertible rule. An invertible rule may be applied ("backwards") at any point in the procedure, whereas care must be taken to apply a non-invertible rule only if we either know (on special grounds) that the premisses are valid if the conclusion is, or hedge our bets by attempting or being prepared to attempt other proofs as well.

The classical system has the property that every rule is invertible. Hence (assuming the system to be complete, as in fact it is) finding a proof of a classically valid sequent is

[2] Note that the axioms of GF do not rule out e.g. an infinitely descending sequence $a_0,a_1,..$ such that $f(a_{i+1})=a_i$ for all i.

just a matter of plodding along, applying rules in whatever order as long as the procedure is sufficiently systematic. If the contraction in $\forall\rightarrow$ and $\rightarrow\exists$ and the identity rules is removed, these rules will no longer be invertible, but all the other rules are. Hence these other rules require no contraction.

The intuitionistic system is more complicated. If we remove the uses of contraction, we find that there are eight non-invertible rules: the four identity rules and the rules $\supset\rightarrow$, $\forall\rightarrow$, $\rightarrow\vee$, $\rightarrow\exists$. Thus we know that these rules (but not the others) may require contraction.The examples of §3 establish that contraction is necessary, except for $\rightarrow\exists$ and $\rightarrow\vee$, where contraction is syntactically impossible in the present formulation of the intuitionistic sequent calculus. In the case of the $\supset\rightarrow$-rule, the premiss $B,\Gamma\rightarrow C$ is logically valid if the conclusion is, so contraction is needed only in the other premiss. However, even with the introduction of contraction, the rules $\rightarrow\vee$, $\supset\rightarrow$, $\rightarrow\exists$ remain non-invertible. Hence (assuming the system to be complete, as will be proved below), the task of constructing proofs (from the bottom up) is a more delicate one in the intuitionistic than in the classical case. We know indeed that an invertible rule may be applied at any point - e.g. $\rightarrow\supset$ can be applied at any time - but in facing a choice between e.g. $\supset\rightarrow$ and $\rightarrow\vee$ we must either apply special considerations or try both ways.

6 The Heyting interpretation

The standard way of thinking in and about intuitionistic logic is to use some form of the Heyting interpretation. For a sound practical grasp of intuitionistic logic, it is quite sufficient to learn to use this interpretation in a schematic and informal way now to be explained. First a point concerning identity in intuitionistic logic must be dealt with.

Functions in intuitionistic logic are always computable functions. That is, for a function f to be defined, there must be defined some effective procedure for computing the value f(x) given an input x. In the present context, the input is always some kind of finite data structure. The computability of functions means that given objects x and y and a function f we must always be able to decide whether f has the value y applied to x, i.e. whether f(x) is identical with y. Since the identity function (mapping x to x) is certainly computable, this means that we must, for given x,y, be able to decide whether or not x is identical with y. The relevant sense of identity is not, however, that expressed by = in intuitionistic predicate logic. There is no assumption is intuitionistic logic that there is any method for deciding whether or not x=y is true. The decidable identity is sometimes called strict identity or definitional identity. = usually stands for some relation of extensional identity, the exact definition of which will vary depending on what kind of objects are under discussion. In special cases = and definitional identity may coincide. = is always assumed to satisfy the usual axioms and rules for identity.

An example should clarify this distinction. Two algorithms are extensionally identical if they give the same output for every input. This is not a decidable relation. They are definitionally or strictly (or "intensionally") identical if they consist of precisely the same instructions, perhaps after both algorithms have been reduced to some primitive notation. This is a decidable relation, i.e. given two expressions A and B we can decide by inspection whether or not they formulate the same algorithm in this sense.

Now the basic point in the Heyting interpretation (and indeed in intuitionistic thinking generally) is that the meaning of a statement is explained by saying what is meant by a proof of that statement. Thus assume that each predicate symbol p (and in particular the identity symbol =) is associated with some notion of what constitutes a proof of a statement of the form $p(t_1,..,t_n)$. We then have the following schematic explanations of statements expressed in predicate logic:

$\perp$ has no proof.
A proof of A&B is a pair consisting of a proof of A and a proof of B.
A proof of $A \vee B$ is a proof of A or a proof of B.
A proof of $A \supset B$ is a function taking proofs of A to proofs of B.
A proof of $\exists x A(x/\alpha)$ is a pair consisting of a term t and a proof of $A(t/\alpha)$.
A proof of $\forall x A(x/\alpha)$ is a function taking each t to a proof of $A(t/\alpha)$.

To say that a sequent $A_1,..,A_n \rightarrow B$ is intuitionistically valid is to say that we have a schematic method (independent of what is meant by a proof of an atomic statement) which yields a proof of B when it is applied to proofs of $A_1,..A_n$. It is presupposed that the interpretation of = is such as to validate the usual rules.

To see how this is used, let us verify that the intuitionistic $\rightarrow\supset$-rule is intuitionistically correct, but not the classical $\rightarrow\supset$-rule, and similarly for the $\rightarrow\forall$-rule.

Suppose $A,\Gamma \rightarrow B$ is a valid sequent. This means that we have a method M which yields a proof of B when it is applied to proofs of the formulas in A,Γ. For the sequent $\Gamma \rightarrow A \supset B$ to be valid, we need a method which yields a proof of $A \supset B$ when it is applied to proofs of the formulas in Γ, a proof of $A \supset B$ being a function which yields a proof of B when applied to a proof of A. Such a method M' is defined as follows: given proofs of the formulas in Γ, let f be the function which applied to a proof of A yields the proof of B which we obtain from the application of M to the proof of A and the given proofs of the formulas in Γ.

There is no obstacle to introducing intuitionistic sequents with more than one formula in the consequent. The variant of the intuitionistic system obtained by allowing sequents $\Gamma \rightarrow \Delta$ with more than one formula in the consequent differs from the classical system in two rules. Instead of the classical $\rightarrow\supset$-rule and $\rightarrow\forall$-rule, the intuitionistic system has the rules

$$\frac{A,\Gamma \rightarrow B}{\Gamma \rightarrow \Delta, A \supset B}$$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow \Delta, \forall x A(x/\alpha)}$$

In addition, the use of contraction in the $\rightarrow\exists$-rule is not necessary in the intuitionistic system. Note that these two rules are not invertible. (Also, the $\supset\rightarrow$-rule remains non-invertible in the system.) This variant of the intuitionistic sequent calculus occurs in [3] and in [4] (in the form of a tableau system).

A sequent $\Gamma \rightarrow \Delta$ is intuitionistically valid if there is a method which applied to proofs of the formulas in Γ yields a proof of some formula in Δ. The classical $\rightarrow\supset$-rule will not be logically valid, however. That $A,\Gamma \rightarrow \Delta,B$ is valid means that we have a method M which applied to proofs of the formulas in A,Γ yields a proof of a formula in Δ or a proof of B. For $\Gamma \rightarrow \Delta, A \supset B$ to be valid we need a method M' which applied to proofs of the formulas in Γ yields either a proof of a formula in Δ or a function taking proof of A to proofs of B. But M provides us with no such method. The previous reasoning yields only a function f which applied to a proof of A gives either a proof of B or a proof of a formula in Δ.

Again, suppose $\Gamma \rightarrow A$ is intuitionistically valid. We then have a schematic method which yields a proof of A applied to proofs of the formulas in Γ. Since the method is schematic, we may substitute t everywhere for α, obtaining a proof of $A(t/\alpha)$ when proofs of the formulas in Γ are given. But then we have a method for obtaining a proof of $\forall x A(x/\alpha)$ given proofs of the formulas in Γ. If $\Gamma \rightarrow \Delta, A$ is valid we cannot, however, conclude that $\Gamma \rightarrow \Delta, \forall x A(x/\alpha)$ is valid.

To make the Heyting interpretation formally precise, and to use it in giving formal counterexamples, is a difficult task, in particular if one wants an intuitionistically acceptable theory (see [3] for a discussion of the issues involved). In the present context, and often in practice, the chief role of the Heyting interpretation is as an informal and intuitive way of understanding intuitionistic logic. For formal results, the very different interpretation introduced by Kripke is much easier to deal with, and a completeness proof for intuitionistic logic relative to Kripke semantics will be given below.

7 Kripke semantics

A Kripke structure $I = \langle W, R, O, \Vdash \rangle$ consists of a set W of objects picturesquely referred to as states of information, a reflexive and transitive relation R over W, a function O assigning a non-empty set O(K) of objects to each K in W, and a relation $\Vdash$ described below. For vividness, R(a,b) is thought of as "b is a possible extension of the information in a". Some further conditions on I will be stated below.

K* will be used as a variable over R-extensions of K, so that "for every K*,..K*.." means "for every K', if KRK' then ..K'..", and similarly for existential statements.

A *K-term* is an expression $s(a_1,...a_n)$ obtained by substituting members a_i of O(K) for each parameter α_i in a term $s(\alpha_1,...\alpha_n)$. (Thus in particular the members of O(K) are K-terms.) s and t will also be used to denote K-terms. A *K-formula* is the result $A(t_1,..t_n)$ of substituting K-terms t_i for the parameters α_i in a formula $A(\alpha_1,..\alpha_n)$.

The remaining conditions on I can now be stated. O is assumed to be R-monotone: $O(K) \subseteq O(K^*)$. $\Vdash$ is a relation between states of information K and atomic K-formulas which is also R-monotone, i.e. if $K \Vdash A$ then $K^* \Vdash A$. We read $K \Vdash A$ as "K forces A". $\Vdash$ is thought of as a consequence relation: the information in K constructively necessitates the statement A. Accordingly it is assumed that no K has the relation $\Vdash$ to $\perp$ and that the identity relation is appropriately forced: $K \Vdash s=s$ for all K-terms s; if $K \Vdash s=t$ and $K \Vdash A(s)$ then $K \Vdash A(t)$, for all atomic A.

The relation $K \Vdash A$ is extended to arbitrary K-formulas A by the following clauses:

$K \Vdash A \& B$ iff $K \Vdash A$ and $K \Vdash B$

$K \Vdash A \vee B$ iff $K \Vdash A$ or $K \Vdash B$

$K \Vdash A \supset B$ iff for all K*, if $K^* \Vdash A$ then $K^* \Vdash B$

$K \Vdash \forall x A(x)$ iff for all K* and all K*-terms t, $K^* \Vdash A(t)$

$K \Vdash \exists x A(x)$ iff for some K-term t, $K \Vdash A(t)$

An easy induction establishes that the extended $\Vdash$-relation is still R-monotone. It is also easily established that if $K \Vdash s=t$ and $K \Vdash A(s)$ then $K \Vdash A(t)$, for every formula A.

The logical consequence relation can now be defined. Let $\alpha_1,..\alpha_n$ be a list containing all parameters occurring in A or in some formula in Γ. The sequent $\Gamma \to A$ is valid in I if and only if for all $K \in I$ and all K-terms $t_1,..t_n$, if K forces $B(t_1,..t_n/\alpha_1,..\alpha_n)$ for all B in Γ, then K forces $A(t_1,..t_n/\alpha_1,..\alpha_n)$. $\Gamma \to A$ is logically valid, and A is a logical consequence of Γ, if and only if $\Gamma \to A$ is valid in all structures I.

It is a simple exercise to verify the

Kripke soundness of GI: Every sequent provable in GI is valid.

The converse of this, i.e. the completeness of GI will not be proved here, but follows by a somewhat simpler variant of the completeness proof for GF.

Any set of classical structures can be identified with a Kripke structure in which R is the identity relation, and classical validity amounts to the same as Kripke validity in all such structures. In terms of the informal reading of the definitions given above, classical states of information are *complete:* any atomic fact not necessitated by a state of information K is ruled out by K in the sense of not being necessitated by any possible extension of K (a "closed world assumption").

I is a *quasi-free* Kripke structure if the obvious translations of the conditions defining a quasi-free structure (see §4) hold, which is to say that the identity rules and axioms of GF hold in I. We thus have the

Kripke soundness of GF: Every sequent provable in GF is valid in all quasi-free Kripke structures.

It is a simple matter to verify the semantical versions of the properties of GF enumerated at the end of §4. As an example, take the semi-invertibility of the $\supset\to$-rule: if $A \supset B, \Gamma \to C$ is valid in all Kripke structures, then so is $B, \Gamma \to C$, since $K \Vdash A \supset B$ whenever $K \Vdash B$. Thus a completeness proof for GF w.r.t. Kripke semantics yields the invertibility of rules etc. as corollaries.

8 Kripke completeness of GF

Some words of explanation may be helpful for those who are not familiar with (classical) completeness proofs for intuitionistic logic. The following completeness proof for GF (modelled on the exposition in [4] of Beth-Kripke-Smullyan-Hintikka type completeness proofs for somewhat different formulations of intuitionistic logic) should be compared with the completeness proof for the classical cut-free sequent calculus given in [9]. In that proof, one attempts to construct a derivation of a given sequent, utilizing the fact that all rules are invertible. The result is either a proof tree or a tree with a branch (in general infinite) which does not terminate in an axiom and can be used to define a counterexample. In the intuitionistic case, the attempt to construct a proof proceeds in the same way, except that the construction must take alternatives into account whenever a non-invertible rule is tried, so that a class of attempted proof trees is constructed. If the sequent is not provable, each of these trees yields an infinite or finitely failed branch which is taken to be a state of information in a Kripke counterexample. The argument differs somewhat from that of [9] in that the sequent is assumed at the outset to be unprovable, and only the branches used to define the counterexample are considered in the argument. (For this reason no appeal to König's lemma is necessary.) Some complications are introduced by the identity rules. The concept of model set defined below (following the exposition in [4]) is only an ad hoc notion designed to make the structure of this particular completeness proof clearer.

A *signed formula* is an expression TA or FA where A is a formula. If K is a set of signed formulas, $\tau(K)$ denotes the set of terms formed (using function symbols) from parameters occurring in formulas in K. Similarly $\tau(P)$ if P is a set of parameters. A *K-instance* of a formula B is B itself or the result $B(s_1/\alpha_1)..(s_n/\alpha_n)$ of a series of successive substitutions, where $\alpha_1,..\alpha_n$ are different parameters and for each i, $Ts_i=\alpha_i \in K$ or $T\alpha_i=s_i \in K$ and α_i does not occur in s_i. Now suppose S is some reflexive relation between sets of signed formulas such that the following three conditions are satisfied whenever KSK' holds:

(S1) $\tau(K)\subseteq\tau(K')$

(S2) For all universally quantified A, if $TA\in K$ then $TA'\in K'$ for some K-instance A' of A.

(S3) For all implications $A\supset B$, if $TA\supset B\in K$ then either $TB'\in K$ for some K-instance B' of B, or $TA'\supset B'\in K'$ for some K-instance $A'\supset B'$ of $A\supset B$.

Now let $\mathbb{C}$ be a family of sets K of signed formulas. We define a Kripke structure as follows. W is $\mathbb{C}$ and the relation R is the transitive closure of the relation S (restricted to $\mathbb{C}$). O(K) is the set, assumed to be non-empty, of parameters occurring in formulas in K. Thus the K-terms are the terms in $\tau(K)$ and the K-formulas are the formulas containing only terms from $\tau(K)$ (equivalently, containing only parameters taken from O(K)). For atomic K-formulas A, finally, define inductively $K \Vdash A$ to hold if $TA \in K'$ for some K' such that K'RK, or A is A(s) where $K \Vdash A(t)$ and $K \Vdash s=t$. It is easily seen that this yields a Kripke structure provided it holds for all K that $T\bot \notin K$ and $Ts=s \in K$ for every $s \in \tau(K)$.

In the following arguments I will use without special mention the clauses defining the forcing relation for non-atomic formulas, the R-monotonicity of the forcing relation - often in the form "if KRK' and K' does not force A then K does not force A" - and the substitution principle: if $K \Vdash s=t$ and $K \Vdash A(s)$ then $K \Vdash A(t)$, for every formula A. As a consequence of the substitution principle, if A' is a K-instance of A, $K^* \Vdash A$ iff $K^* \Vdash A'$ for every K*.

Now let $\mathbb{C}$, S and $\Vdash$ be as stated above. $\mathbb{C}$ is a *model set* iff in addition to the conditions $T\bot \notin K$ and $Ts=s \in K$ for every $s \in \tau(K)$ the following holds for all $K \in \mathbb{C}$ and all A,B:

(H1) If $K \Vdash A$ then $FA \notin K$, for atomic A.

(H2) If $TA\&B \in K$ then $TA \in K$ and $TB \in K$.

(H3) If $FA\&B \in K$ then $FA \in K$ or $FB \in K$.

(H4) If $TA \vee B \in K$ then $TA \in K$ or $TB \in K$.

(H5) If $FA \vee B \in K$ then there are K' and K" and a K-instance $A' \vee B'$ of $A \vee B$ such that KSK', KSK", $FA' \in K'$ and $FB' \in K''$.

(H6) If $TA \supset B \in K$ then for some K-instance $A' \supset B'$ of $A \supset B$, either $TB' \in K$ or for some K', KSK' and $FA' \in K'$.

(H7) If $FA \supset B \in K$ then $TA \in K$ and $FB \in K$.

(H8) If $T\forall xA(x) \in K$ then for every $t \in \tau(K)$ there is a K-instance $\forall xA'(x)$ of $\forall xA(x)$ and a term $s \in \tau(K)$ such that $K \Vdash s=t$ and $TA'(s) \in K$.

(H9) If $F\forall xA(x) \in K$ then $FA(\alpha) \in K$ for some $\alpha \in \tau(K)$.

(H10) If $T\exists xA(x) \in K$ then $TA(\alpha) \in K$ for some $\alpha \in \tau(K)$.

(H11) If $F\exists xA(x) \in K$ then for every $t \in \tau(K)$ there is a K-instance $\exists xA'(x)$ of $\exists xA(x)$, a term $s \in \tau(K)$, and a K' such that KSK' and $K \Vdash s=t$ and $FA'(s) \in K'$.

We have the following

Model lemma: Suppose $\mathbb{C}$ is a model set. For all formulas A and all K in $\mathbb{C}$: if $TA \in K$ then K forces A, and if $FA \in K$ then K does not force A.

Proof: By induction on A. All cases are equally simple except for $T\supset$ and $T\forall$. For the implication case, first note that if $TA \supset B \in K$ and $K \Vdash A$ then $K \Vdash B$ by (H6) and the induction hypothesis. We must show that if $TA \supset B \in K$ and $K^* \Vdash A$ then $K^* \Vdash B$.First suppose KSK^*. By (S3), either $K \Vdash B$, in which case $K^* \Vdash B$, or $TA' \supset B' \in K^*$ for some K-instance $A' \supset B'$ of $A \supset B$; thus if $K^* \Vdash A$ then $K^* \Vdash B$. Now suppose $KSK'RK^*$. Again either $K \Vdash B$, in which case $K^* \Vdash B$, or $TA' \supset B' \in K'$ for some K-instance $A' \supset B'$ of $A \supset B$; in the latter case it follows by induction on R that if $K^* \Vdash A$ then $K^* \Vdash B$.

Now suppose $T\forall xA(x) \in K$. We must show that for all K^* and all $t \in \tau(K^*)$, $K^* \Vdash A(t)$. First suppose KSK^*. By (S2), for some K-instance $\forall xA'(x)$ of $\forall xA(x)$, $T\forall xA'(x) \in K^*$. By (H8) there is a K^*-instance $\forall xA''(x)$ of $\forall xA'(x)$ and a term $s \in \tau(K^*)$ such that $K^* \Vdash s=t$ and $TA''(s) \in K^*$. Thus $K^* \Vdash A''(s)$, so $K^* \Vdash A''(t)$, so $K^* \Vdash A(t)$. Now suppose $KSK'RK^*$. For some K-instance $\forall xA'(x)$ of $\forall xA(x)$, $T\forall xA'(x) \in K'$. Again by induction on R, $K^* \Vdash A'(t)$ for every $t \in \tau(K^*)$, so $K^* \Vdash A(t)$. ■

We are now in a position to prove the

Kripke completeness of GF: Every sequent valid in all quasi-free Kripke structures is provable in GF.

Proof: Given a sequent which is not provable in GF, a quasi-free Kripke structure will be found in which the sequent is not valid. We define ("construct") a class of sequences Σ of unprovable sequents, each Σ with an associated infinite set P_Σ of parameters. At each point in the construction a finite number of the parameters in P_Σ are designated as *inactive* parameters. At various points, new sequences are introduced based on some sequence Σ. These new sequences have Σ as their *parent* sequence; their *ancestors* are Σ and the ancestors of Σ. The reflexive parent relation - i.e. the union of that relation with the identity relation - will be used as the relation S in the model lemma, and the relation R in that lemma will be the reflexive ancestor relation. The set of parameters of a newly introduced sequence is the union of the parameter set of its parent (with the same parameters designated as inactive) and an infinite set of active parameters disjoint from every other set of parameters introduced.

We start with the sequence containing only the given unprovable sequent and with an associated infinite set of parameters including those in the given sequent. All parameters are initially active. To perform a step in the construction, we pick a sequence Σ in accordance with some fair principle by which every sequence will eventually be dealt with. Next we extend that sequence as follows in what will be called the first phase of the stage. Operate on the last sequent of Σ in accordance with the following clauses to obtain a new sequence Σ'; repeat this with Σ' until none of the clauses applies to the last sequent:

If the last sequent is $A\&B,\Gamma \rightarrow C$, add $A,B,\Gamma \rightarrow C$ to Σ.

If it is $\Gamma \rightarrow A\&B$, add $\Gamma \rightarrow A$ to Σ if this sequent is unprovable, otherwise add $\Gamma \rightarrow B$.

If it is $A \vee B,\Gamma \rightarrow C$, add $A,\Gamma \rightarrow C$ if this sequent is unprovable, otherwise add $B,\Gamma \rightarrow C$.

If it is $\Gamma \rightarrow A \supset B$, add $A,\Gamma \rightarrow B$.

If it is $\Gamma \rightarrow \forall x A(x)$, add $\Gamma \rightarrow A(\alpha)$ for some active parameter α that does not occur in Γ or in any previous sequent in Σ or any ancestor of Σ.

If it is $\exists x A(x),\Gamma \rightarrow C$, add $A(\alpha),\Gamma \rightarrow C$ for some active parameter α that does not occur in Γ,C or in any previous sequent in Σ or any ancestor of Σ.

When Σ has been extended as far as possible using these prescriptions, extend the resulting sequence in accordance with the following rule until no equation remains in the antecedent of the last sequent (the second phase):

If the last sequent is $\alpha = s,\Gamma \rightarrow B$ or $s = \alpha,\Gamma \rightarrow B$, add $\Gamma(s/\alpha) \rightarrow B(s/\alpha)$ to S as the last sequent. If s is not identical with the parameter α, designate α as an inactive parameter. If the last sequent is $f(s_1,..s_n)=f(t_1,..t_n),\Gamma \rightarrow C$, add $s_1=t_1,...s_n=t_n,\Gamma \rightarrow C$.

Next (the third phase) perform *one* of the following operations, if any is applicable. Here too it is presupposed that some fair principle is applied, whereby every (occurrence of a) formula will eventually be dealt with, and in the case of $\forall x A(x)$ that we will return infinitely many times to the formula.

If the last sequent of Σ is $A \supset B,\Gamma \rightarrow C$, add $B,\Gamma \rightarrow C$ to Σ as the last sequent if this sequent is unprovable, otherwise introduce a new sequence starting with $A \supset B,\Gamma \rightarrow A$.

If the last sequent is $\Gamma \rightarrow A \vee B$, introduce new sequences starting with $\Gamma \rightarrow A$ and with $\Gamma \rightarrow B$.

If the last sequent is $\Gamma \rightarrow \exists x A(x)$, introduce new sequences starting with $\Gamma \rightarrow A(s)$ for every term s containing only active parameters.

If the last sequent is $\forall x A(x),\Gamma \rightarrow B$, add $\forall x A(x),A(t),\Gamma \rightarrow B$ to Σ as last sequent, where t is the first term with parameters from P_Σ which has not already been used for instantiating $\forall x A(x)$, provided all parameters in t are currently active. If t contains currently inactive parameters $\alpha_1,..\alpha_n$, add instead the sequent $\forall x A(x),A(s),\Gamma \rightarrow B$, where s is obtained from t by making successive substitutions $s_1,..s_n$ for $\alpha_1,..\alpha_n$, the terms s_i being chosen so that $\alpha_i=s_i$ or $s_i=\alpha_i$ occurs in the antecedent of some sequent in Σ or some ancestor of Σ and for $i=1,..n$, α_i was made inactive through the substitution s_i/α_i, in the indicated order.

This concludes the description of one stage in the construction. The final sequences Σ are those obtained by iterating the above steps ad infinitum. For each Σ, let K_Σ be the set of signed formulas containing TA for every A that occurs in the antecedent of some sequent in Σ, Ts=s for every term s with parameter from P_Σ, and FA for every A that occurs as the consequent of some sequent in Σ. We obtain a Kripke structure by defining the relation S as indicated above. To see this, we must check that (S1)-(S3) hold. This is clear from the fact that implications and universal quantifications in the antecedent are retained when a child sequence to Σ is introduced, although possibly after a number of successive substitutions. Also, $Ts=s\in K_\Sigma$ by definition. Finally, $\perp$ cannot occur in the antecedent of any unprovable sequent.

To conclude the proof we need to establish that this structure is in fact a model set and also that it is quasi-free.

To begin with the simplest part, the properties (H2),(H3),(H4),(H7),(H9),(H10) hold since the corresponding formulas will be dealt with in the first phase of the stage at which they first appear. Similarly (H5),(H6),(H8),(H11) follow from the clauses defining the third phase, taking into account that substitutions may have previously taken place in the second phase. (H11) and (H8) involve equalities. For (H11), suppose that the term t contains parameters from the list $\alpha_1,..\alpha_n$ of parameters inactive at the stage when $\Gamma\rightarrow\forall xA(x)\in\Sigma$ is dealt with. There are terms $s_1,..s_n$ such that $\alpha_i=s_i$ or $s_i=\alpha_i$ occurs in the antecedent of some sequent in Σ or some ancestor of Σ, for $i=1,..n$, and the parameters $\alpha_1,..\alpha_n$ were rendered inactive by substitutions $s_1/\alpha_1...s_n/\alpha_n$, in that order. The term s obtained by successive substitutions $t(s_1/\alpha_1)...(s_n/\alpha_n)$ will contain no inactive parameters and so Σ will have a child beginning with $\Gamma\rightarrow A(s)$, and K_Σ forces s=t by the substitution principle.

For (H1) and to show that the structure is quasi-free we must consider atomic formulas. By the definitions of K_Σ and of forcing in model sets, $K_\Sigma \Vdash A$ for atomic A iff (1) A is s=s where $s\in\tau(P_\Sigma)$, or (2) A occurs in the antecedent of some sequent in Σ or some ancestor of Σ, or (3) A is A(s) where $K_\Sigma \Vdash A(t)$ and $K_\Sigma \Vdash s=t$. We need a more illuminating characterization.

Let aa(Σ) denote the set of atomic formulas that occur in the antecedent of some sequent in Σ or in some ancestor of Σ. Let $\alpha_1,\alpha_2,..$ be the parameters which are rendered inactive through the substitutions $s_1/\alpha_1,s_2/\alpha_2,..$ (in this order) at some stage in the construction of Σ and its ancestors. Thus $s_i=\alpha_i$ or $\alpha_i=s_i$ belongs to $aa(\Sigma)$. The term t' is a Σ-*variant* of t if t' can be obtained from t by successive substitutions in which some occurrences of α_i are replaced by s_i or conversely. Similarly for formulas A and A'. This is clearly an equivalence relation. If t' is is obtainable from t by substitutions involving only $\alpha_{j_1},\alpha_{j_2},..\alpha_{j_n}$ and $s_{j_1},s_{j_2},..s_{j_n}$ where $j_1<...<j_n$, t' and t are Σ-variants *via* $\alpha_{j_1},\alpha_{j_2},..\alpha_{j_n}$.

We can now state the following

Lemma 1: $K_\Sigma \Vdash s=t$ iff s and t are Σ-variants. For all atomic formulas A other than equations, $K_\Sigma \Vdash A$ iff $A'\in aa(\Sigma)$ for some Σ-variant A' of A.

To prove this, first consider equations s=t. The case (1) is immediate. For (2) we must show that if $s=t\in aa(\Sigma)$, then Σ and t are Σ-variants. Every equation in the antecedent of a sequent will be disposed of in the second phase of some stage (depending on the sequent): we argue by induction on the number of steps required in that phase to eliminate s=t. So let $s=t,\Gamma\rightarrow C$ be a sequent in Σ or some ancestor of Σ and consider the second phase of the stage in which s=t is dealt with. If s=t is not dealt with first in that phase, either s=t are unchanged after the first step and hence Σ-variants by the induction hypothesis, or else $s(s_j/\alpha_j)$ and $t(s_j/\alpha_j)$ are Σ-variants for some α_j,s_j, whence s and t are again Σ-variants. Thus we may assume that s=t is dealt with first. There are two possibilities. If s=t is $f(s_1,..s_n)=f(t_1,..t_n)$, the induction hypothesis yields that s_i and t_i are Σ-variants for i=1,..n, so s and t are Σ-variants. If one of s and t is an α_i, which is the second possibility, it is immediate that s and t are Σ-variants. For (3), finally, we must show that if u(t) and v(t) are Σ-variants and s and t are Σ-variants, then u(s) and v(s) are Σ-variants. This follows directly from the definition of Σ-variants.

Now the second part of the lemma. The cases (1) and (2) are immediate. For (3), suppose some Σ-variants A' of A(t) and s'=t' of s=t belong to $aa(\Sigma)$. By the first part, s and t are Σ-variants, so A' and A(s) are Σ-variants. This concludes the proof of the lemma.

We also need

Lemma 2: Suppose t and t' are Σ-variants via $\alpha_{j_1},\alpha_{j_2},..\alpha_{j_n}$. Then for $k\geq j_n$, the terms $t'(s_1/\alpha_1)...(s_k/\alpha_k)$ and $t(s_1/\alpha_1)...(s_k/\alpha_k)$ coincide, and similarly for formulas.

To prove this, it suffices to consider the case where t' is obtained from t by substituting s_p for some occurrences of α_p. (By the symmetry of Σ-variance, substitutions of α_p for some occurrences of s_p are covered by the same argument.) Since s_p does not contain $\alpha_1,..\alpha_{p-1}$, $t'(s_1/\alpha_1)...(s_{p-1}/\alpha_{p-1})$ yields the same result as substituting s_p for the corresponding occurrences of α_p in $t(s_1/\alpha_1)...(s_{p-1}/\alpha_{p-1})$. The substitution (s_p/α_p) thereupon smothers the effect of the previous substitution of s_p, yielding a result that coincides with $t(s_1/\alpha_1)...(s_p/\alpha_p)$, since α_p does not occur in s_p. By induction on the number of substitutions leading from t to t', this proves Lemma 2.

We can now establish the property (H1). Suppose A is atomic and $FA \in K_\Sigma$. This means that there is a sequent $\Gamma \rightarrow A \in \Sigma$. We must show that K_Σ does not force A. First suppose A is s=t. If K_Σ forces s=t, s and t are Σ-variants by Lemma 1. But then by Lemma 2, Σ will contain an identity axiom. If A is not an equation and K_Σ forces A, there is a sequent $A',\Delta \rightarrow B$ in Σ or in some ancestor of Σ, where A' is a Σ-variant of A. But then Lemma 2 shows that Σ will contain a logical axiom. This is the part of the argument in which we use the fact that inactive parameters will never reappear in Σ once they have been eliminated by a substitution: we need to know that the substitutions by which A and A' (or t and t') can be made to coincide are in fact carried out in Σ.

It only remains to verify that the structure is quasi-free. For this we need to show (i) if $f(s_1,..s_n)$ and $f(t_1,..t_n)$ are Σ-variants, then s_i and t_i are Σ-variants for i=1,..n; (ii) $f(s_1,..s_n)$ and $g(t_1,..t_m)$ are not Σ-variants; (iii) t and s are not Σ-variants if t is a proper subterm of S. These properties are easily established using Lemma 2. ■

9 Classical vs intuitionistic logic

It was noted in §6 that the intuitionistic sequent calculus is computationally more complicated than the classical system because of the non-invertibility of the rules $\rightarrow\vee$, $\rightarrow\exists$, $\supset\rightarrow$. In the classical system all rules are invertible when contraction is incorporated into $\forall\rightarrow$ and $\rightarrow\exists$ as is done in GC. Intuitionistically we must use contraction with respect to the rules $\supset\rightarrow$ and $\forall\rightarrow$, and the non-invertible rules must in general be applied in a particular order for a proof to be found. Intuitionistic sequents have only one formula in the consequent, but one would not expect the computational advantages of this to outweigh the complications introduced by the greater subtlety of intuitionistic logic.[3] Experiments in propositional logic support this impression (see [5]). The question therefore naturally arises: why should we try to base a logic programming system on the intuitionistic rather than the classical sequent calculus?

Intuitionistic logic has the much-advertised disjunction and existence properties, i.e.if

[3] It is also notable in this context that classical logic is interpretable in intuitionistic logic, but not conversely.

a disjunction $A \vee B$ is intuitionistically provable, then A or B is provable, and if $\exists x A(x)$ is provable, some instance $A(t)$ is provable. As a consequence, it is always possible to extract from an intuitionistic predicate-logic proof of $\exists x A(x)$ a term t and a proof of $A(t)$. Various such extraction procedures are treated in the proof-theoretical literature. When a cut-free sequent calculus is used, the extraction is of the simplest possible kind: inspection of the rules reveals that a proof of a sequent of the form $\rightarrow A \vee B$ must end with an application of $\rightarrow\vee$, and hence contain a proof of either $\rightarrow A$ or $\rightarrow B$. Similarly for existence formulas. When other formalisms are used, the procedures are more complex (normalization, cut elimination, functional interpretation).

The disjunction and existence properties hold also for various formalized mathematical theories in their intuitionistic interpretation, for example first order arithmetic. Generally speaking, one expects these properties to hold whenever a statement is proved outright using constructive reasoning. But of course the properties do not hold for general derivations from assumptions. That $\Gamma \rightarrow A \vee B$ is provable does not imply that $\Gamma \rightarrow A$ or $\Gamma \rightarrow B$ is provable, the simplest counterexample being $p \vee q \rightarrow p \vee q$. Similarly, $\exists x p(x) \rightarrow \exists x p(x)$ is provable, but not $\exists x p(x) \rightarrow p(t)$ for any term t.

The use of intuitionistic logic for computational purposes has occasionally been proposed with reference to the disjunction and existence properties, the general idea being that since we seek explicit solutions and instances in logic programming, it makes good sense to use a logic that provides such explicit solutions.

This argument is not very convincing, however. First, there is the obvious point that the validity of the extended disjunction and existence properties

(D) if $\Gamma \rightarrow A \vee B$ is provable then $\Gamma \rightarrow A$ or $\Gamma \rightarrow B$ is provable

(E) if $\Gamma \rightarrow \exists x A(x)$ is provable then $\Gamma \rightarrow A(t)$ is provable for some t

depends on the allowable programs Γ. A second point, which is more easily overlooked, is that intuitionistic logic imposes restrictions that go far beyond the disjunction and existence properties. For example, the sequent

$$\forall x(\neg p(x) \supset q(x)), \neg q(a) \rightarrow \exists x p(x)$$

is not intuitionistically valid, although the explicit instance $p(a)$ follows from the premisses using classical logic.

Nor is there any clear sense in which the use of intuitionistic logic helps us find explicit instances, even when such instances follow intuitionistically. The idea is that we find an explicit solution of $A(x)$ by proving $\exists x A(x)$ and then extracting a solution from this proof. In resolution logic for Horn clauses there is a very useful procedure for doing this, viz. by assuming $\forall x \neg A(x)$ and deriving $\perp$. This indeed is reasoning using classical logic, but the essential point is that in the special case of Horn clauses, we can always extract an explicit solution from this derivation (and transform the derivation into a constructively valid proof of an instance $A(t)$). If we go beyond Horn clauses the situation is very different. Consider the type of proof procedure available for the sequent calculus.[4] In seeking to construct an intuitionistic proof of the sequent $\rightarrow \exists x A(x)$ we introduce a dummy variable X and continue to operate on the expression $A(X)$, attempting at various points to perform substitutions for X so as to make the whole construction into a proof. But this procedure is equally applicable in the case of the classical sequent calculus as a means of seeking explicit provable instances (including the intuitionistically valid instances).

Thus there is no obvious point in introducing intuitionistic logic at all: it provides fewer explicit solutions than classical logic, there is no known intuitionistic method for extracting explicit solutions which is not equally applicable in classical logic, and finally, intuitionistic logic is computationally even more difficult than classical logic. Of course these objections to the use of intuitionistic logic fall away if we take the view that classical proofs of explicit instances are simply not valid, but must be replaced by constructive proofs. This frankly philosophical justification for using intuitionistic logic will not be considered here.

A better procedure for establishing the disjunction and existence properties is in my opinion to *stipulate* that (D) and (E) hold as part of the definition of the syntax and semantics of a logic programming language. That is, we understand disjunction and existential quantification as constructions applicable only to *goals,* with the stipulation that a goal $\exists x G(x)$ is fulfilled by fulfilling $G(t)$ for some t, and similarly for disjunctions. We can now make the observation that a fragment of intuitionistic logic provides a natural system of rules for this use of disjunction and existence. An easy argument proves the following well-known

Constructivity lemma. Let **F** be the set of formulas not containing any disjunction or existential formula as a positive subformula, and let **Q** be the set of formulas not containing any disjunction or existential formula as a negative subformula.[5] Then all

[4] See [7] and variants of the procedure in [1] and [6].

[5] Positive and negative subformulas, it will be recalled, are defined as follows:
C is a positive subformula of C.
If $A\&B$ or $A \vee B$ is a positive (negative) subformula of C, then so are A and B.
If $A \supset B$ is a positive (negative) subformula of C, then A is a negative (positive) subformula of C and B is a positive (negative) subformula of C.
If $\exists x A(x)$ or $\forall x A(x)$ is a positive (negative) subformula of C, then so is $A(t)$, for every term t.

sequents occurring in a proof of $\Gamma \rightarrow A$ with $\Gamma \subseteq \mathbb{F}$ and $A \in \mathbb{Q}$ have the disjunction and existence properties.

Proof: Let GI~ be the system GI with the rules $\vee\rightarrow$ and $\exists\rightarrow$ removed and with axioms restricted to $\Gamma \rightarrow \perp$ and $A,\Gamma \rightarrow A$ for formulas A other than disjunctions and existential formulas. Using induction on proofs we find that (D) and (E) hold in full generality for provability in GI~. An induction on formulas shows that no disjunction or existentially quantified formula can appear in the antecedent in any sequent in a proof of $\Gamma \rightarrow A$ if $\Gamma \subseteq \mathbb{F}$ and $A \in \mathbb{Q}$; hence $\Gamma \rightarrow A$ is provable in GI iff it is provable in GI~. ∎

More graphically, the classes $\mathbb{F}$ and $\mathbb{Q}$ can be described as follows, using F, F_i and Q, Q_i as variables over $\mathbb{F}$ and $\mathbb{Q}$ and A as a variable for atomic formulas (including $\perp$):

$$F := A \mid F_1 \& F_2 \mid Q \supset F \mid \forall x F(x)$$

$$Q := A \mid Q_1 \vee Q_2 \mid Q_1 \& Q_2 \mid F \supset Q \mid \forall x Q(x) \mid \exists x Q(x)$$

This class of program statements and goals is considered in [11] (see also references given in [11]).

Granted that GI~ is a natural system of rules in a logic programming system (regarded e.g. as an extension of Prolog), it remains to give a coherent interpretation of the system. Here indeed, intuitionistic logic provides one such interpretation: program statements and goal statements are all read as intuitionistically interpreted predicate logic formulas. It is not clear, however, that this interpretation is at all useful in a logic programming context.

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